The Dirac–Witten operator on pseudo-Riemannian manifolds

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THE DIRAC-WITTEN OPERATOR ON PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. We study the Dirac-Witten operator for spacelike spin submanifolds in pseudo-Riemannian manifolds. When the normal bundles are spin and odd-dimensional, we derive new eigenvalue lower bounds. We also prove the generalized positive mass theorem using this operator.

1. Introduction

In this paper, we will study the Dirac-Witten operator of spacelike spin submanifolds in pseudo-Riemannian manifolds. In the case of spacelike hypersurfaces in 4-dimensional spacetimes, this operator was introduced by Witten to provide a new proof of the positive mass conjecture, which was originally proved by Schoen and Yau using a geometric analysis approach [23, 24, 25, 26, 21]. Inspired by Witten's proof, the eigenvalues of Dirac-Witten-type operators were estimated in [27, 20, 17, 18, 11, 19].

We first study the eigenvalues of the Dirac-Witten operator for spacelike spin submanifolds with spin normal bundles. These could be viewed as the pseudo-Riemannian or the higher codimensional analogue of the results established in [18, 11, 19]. It is surprising to note that when the dimension of the normal bundle is odd, the Dirac-Witten operator has nice properties and certain interesting eigenvalue lower bounds can be established. Inspired by [14, 13], we introduce a local boundary condition for this Dirac-Witten operator for submanifolds with boundary. We obtain eigenvalue lower bounds under this boundary condition when the boundary is a generalized future/past apparent horizon. Next, we study the generalized positive mass theorem for higher dimensional spacetimes with multi-time components. From [26], we know that the Dirac-Witten operator is closely related to the total energy and the total linear momentum 3-vector in 4-dimensional spacetimes. We observe that a similar phenomenon occurs for higher dimensional spacetimes with odd number of time components, and in this case, the total linear momentum is an (n, m)-bivector. Under a generalized dominant energy condition, we close by proving the generalized positive mass theorem.

It is a purely mathematical consideration to define the generalized dominant energy condition, the generalized future/past apparent horizon, as well as the generalized total energy and total linear momentum for higher dimensional spacetimes with multi-time components. They reduce to the standard definitions in general relativity with 1-time component. There are two ways to generalize the Einstein field equations to the case of m-time components by using the tensors $R_{ij} - \frac{R}{2} g_{ij}$ and $R_{ij} - \frac{R}{m} g_{ij}$ respectively. However, no clear relationship can be found in the two cases relating these definitions to the energy-momentum tensor T_{ij}.

2. Submanifolds and Spin Geometry

2.1. **Spacelike submanifolds.** Let N^{n+m} be an $(n+m)$-dimensional pseudo-Riemannian manifold whose metric $\widetilde{g}$ has signature $(\underbrace{1, \cdots, 1}_{n}, \underbrace{-1, \cdots, -1}_{m})$. Let M^n be an n-dimensional spacelike submanifold with the induced Riemannian metric $\overline{g}$. Denote by $\widetilde{\nabla}$ and $\overline{\nabla}$ the Levi-Civita connections of N^{n+m} and M^n respectively. Throughout this paper, we will agree on the following ranges of indices:

$$1 \le \alpha, \beta, \cdots \le n+m; \quad 1 \le i, j, \cdots \le n; \quad n+1 \le A, B, \cdots \le n+m.$$

The Einstein's summation notation is also used.

For any point $p \in M^n$ we consider an orthonormal basis $\{e_\alpha\}$ of T_pN^{n+m} with e_A normal and e_i tangent to M^n. Let $\{\omega^\alpha\}$ be the dual basis of $\{e_\alpha\}$ so that the pseudo-Riemannian metric of N^{n+m} is locally given by

$$\widetilde{g} = \sum_{i=1}^{n} (\omega^i)^2 - \sum_{A=n+1}^{n+m} (\omega^A)^2.$$

The connection 1-forms ω_α^β satisfy

$$\widetilde{\nabla} e_\alpha = e_\beta \otimes \omega_\alpha^\beta.$$

We have

$$\omega_{\alpha\beta} = \widetilde{g}_{\alpha\gamma}\omega_\beta^\gamma = \varepsilon_\alpha \omega_\beta^\alpha, \quad \omega_\alpha = \widetilde{g}_{\alpha\beta}\omega^\beta = \varepsilon_\alpha \omega^\alpha,$$

where $\widetilde{g}_{\alpha\beta}$ are the components of the metric tensor of the manifold N^{n+m} and the ε_α are given by

$$\varepsilon_\alpha = \begin{cases} 1, & \text{for } 1 \le \alpha \le n, \\ -1, & \text{for } n+1 \le \alpha \le n+m. \end{cases} \tag{1}$$

Then the structure equations of N^{n+m} are given by

$$d\omega_\alpha = -\varepsilon_\beta \omega_{\alpha\beta} \wedge \omega_\beta, \qquad \omega_{\alpha\beta} = -\omega_{\beta\alpha},$$
$$d\omega_{\beta\alpha} = -\varepsilon_\gamma \omega_{\beta\gamma} \wedge \omega_{\gamma\alpha} + \frac{1}{2}\varepsilon_\gamma \varepsilon_\delta \widetilde{R}_{\beta\alpha\gamma\delta} \omega_\gamma \wedge \omega_\delta.$$

The curvature tensor $\widetilde{R}_{\alpha\beta\gamma\delta}$, the Ricci tensor $\widetilde{R}_{\alpha\beta}$ and the scalar curvature $\widetilde{R}$ of N are given by

$$\begin{aligned}
\widetilde{R}(X,Y)Z &= \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z, \quad X,Y,Z \in \Gamma(TN);\\
\widetilde{R}_{\alpha\beta\gamma\delta} &= \widetilde{g}(\widetilde{R}(e_\gamma,e_\delta)e_\beta,e_\alpha) = \widetilde{g}_{\alpha\zeta}\widetilde{R^\zeta}_{\beta\gamma\delta};\\
\widetilde{R}ic(X,Y) &= \varepsilon_\alpha\widetilde{g}(\widetilde{R}(e_\alpha,X)Y,e_\alpha),\\
\widetilde{R}_{\alpha\beta} &= \widetilde{R^\gamma}_{\alpha\gamma\beta} = \varepsilon_\gamma\widetilde{R}_{\alpha\gamma\beta\gamma};\\
\widetilde{R} &= \varepsilon_\alpha\widetilde{R}(e_\alpha,e_\alpha) = \widetilde{g}^{\alpha\beta}\widetilde{R}_{\alpha\beta} = \varepsilon_\alpha\varepsilon_\gamma\widetilde{R}_{\alpha\gamma\alpha\gamma}.
\end{aligned}$$

The curvature tensor $\overline{R}_{ijkl}$, the Ricci tensor $\overline{R}_{ij}$ and the scalar curvature $\overline{R}$ of M^n can be defined in a similar way. From submanifold theory, the Gauss, the Ricci and the Codazzi equations for the spacelike submanifold M^n give rise, respectively to

$$\begin{aligned}
\widetilde{R}_{ijkl} &= \overline{R}_{ijkl} + \big(p_{Aik}p_{Ajl} - p_{Ail}p_{Ajk}\big)\\
\widetilde{R}_{ABkl} &= \overline{R}_{ABkl} - (p_{Aik}p_{Bil} - p_{Ail}p_{Bik}) \qquad (2)\\
\widetilde{R}_{iAkj} &= p_{Ajik} - p_{Aikj}
\end{aligned}$$

where $p_{Aij} := \widetilde{g}(\widetilde{\nabla}_i e_A, e_j)$ are the components of the second fundamental form of M^n, and p_{Aijk} the covariant derivative of p_{Aij}, are defined by

$$p_{Aijk}\omega_k = dp_{Aij} - p_{Akj}\omega_{ki} - p_{Aik}\omega_{kj} + p_{Bij}\omega_{BA}.$$

2.2. **Spin connection.** Let $Cl_{n,m}$ be the Clifford algebra with respect to the above pseudo-Riemannian metric $\widetilde{g}$. (We refer to Baum [5, 6] for a detailed algebraic constructions.) For any vector field $X \in \Gamma(TN^{n+m})$,

$$(e_\alpha \wedge e_\beta)X = \widetilde{g}(e_\alpha, X)e_\beta - \widetilde{g}(e_\beta, X)e_\alpha,$$

where $e_\alpha \wedge e_\beta$ is the canonical basis of the Lie algebra $\mathfrak{so}(n,m)$. The Levi-Civita connection on a pseudo-Riemannian manifold N^{n+m} induces the connection on the principle $\mathrm{SO}(n,m)$-bundle with the connection 1-form

$$\omega = -\frac{1}{2}\varepsilon_\alpha\varepsilon_\beta\omega_{\alpha\beta}e_\alpha \wedge e_\beta.$$

The (local) spinorial connection of N^{n+m} is lifted form the principle $\mathrm{SO}(n,m)$ connection as

$$\widetilde{\nabla}\varphi = d\varphi - \frac{1}{4}\varepsilon_\alpha\varepsilon_\beta\omega_{\alpha\beta}e_\alpha e_\beta\varphi.$$

Suppose that M^n is a spin submanifold whose normal bundle ξ in N^{n+m} is also spin. Denote by K the maximal compact subgroup of $\mathrm{Spin}_0(n,m)$, the connected component of the spin group. Let $\$$ be the (local) spinor bundle of N^{n+m}.

Since the following diagram is commutative

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d, "\mathrm{Ad}"] & \mathrm{Spin}_0(n,m) \arrow[d, "\mathrm{Ad}"] \\ \mathrm{SO}(n)\times\mathrm{SO}(m) \arrow[r, "i"] & \mathrm{SO}_0(n,m) \end{tikzcd}$$

the induced spinor bundle $\mathbb{S} := \not{S}|_{M^n}$ is globally defined over M^n.

Denote also by $\widetilde{\nabla}$ and $\overline{\nabla}$ the spin connections on $\mathbb{S}$. It is well-known [5, 6] that there exists a Hermitian inner product $(\ ,\ )$ on $\not{S}$ which is compatible with the spin (local) connection $\widetilde{\nabla}$. Moreover, for any vector field $X \in \Gamma(TN^{n+m})$ and spinor fields $\varphi, \psi \in \Gamma(\not{S})$,

$$(X\varphi, \psi) = (\varphi, X\psi)$$

locally. Note that this inner product is not positive definite and all these data are global over M^n.

The above diagram also implies that, over M^n, there exists a positive definite Hermitian inner product on $\mathbb{S}$ which is defined as

$$\langle\ ,\ \rangle = (\omega\ ,\ )$$

where $\omega = (\sqrt{-1})^{\frac{m(m-1)}{2}} e_{n+1}\cdots e_{n+m}$ is the complex volume element of the normal bundle over M^n. Obviously $\omega^2 = 1$ and we have

$$\big(\omega\varphi, \omega\psi\big) = \big(\varphi, \psi\big),$$

which implies

$$\langle\omega\varphi, \psi\rangle = \langle\varphi, \omega\psi\rangle.$$

Lemma 2.1. *Suppose that M^n is a spin submanifold whose normal bundle is also spin. Over M^n, we have*

(i) *If m is odd, then $\omega e_i = -e_i\omega$, $\omega e_A = e_A\omega$. Moreover,*

$$\langle e_i\varphi, \psi\rangle = -\langle\varphi, e_i\psi\rangle, \qquad \langle e_A\varphi, \psi\rangle = \langle\varphi, e_A\psi\rangle.$$

(ii) *If m is even, then $\omega e_i = e_i\omega$, $\omega e_A = -e_A\omega$. Moreover,*

$$\langle e_i\varphi, \psi\rangle = \langle\varphi, e_i\psi\rangle, \qquad \langle e_A\varphi, \psi\rangle = -\langle\varphi, e_A\psi\rangle.$$

Defining the second fundamental form of M^n by

$$p_{Aij} = \widetilde{g}(\widetilde{\nabla}_i e_A, e_j) = \omega_{jA}(e_i),$$

we have the following spinorial Gauss formula

$$\widetilde{\nabla}_i\varphi = \overline{\nabla}_i\varphi + \frac{1}{2}\omega_{Aj}(e_i)e_Ae_j\varphi = \overline{\nabla}_i\varphi + \frac{1}{2}p_{Aij}e_je_A\varphi.$$

Lemma 2.2. *Suppose that M^n is a spin submanifold whose normal bundle is also spin. For the normal volume element ω, we have*

$$\widetilde{\nabla}_i\omega = p_{Aij}e_je_A\omega.$$

Proof. Note that

$$\widetilde{\nabla}_i e_{n+r} = p_{n+r,ij} e_j - \sum_{1\le s\le m} \omega_{n+s,n+r}(e_i) e_{n+s},$$

we obtain

$$\begin{aligned}
\widetilde{\nabla}_i \omega &= (\sqrt{-1})^{\frac{m(m-1)}{2}} \widetilde{\nabla}_i (e_{n+1} \cdots e_{n+m}) \\
&= (\sqrt{-1})^{\frac{m(m-1)}{2}} \sum_{1\le r\le m} e_{n+1} \cdots \widetilde{\nabla}_i e_{n+r} \cdots e_{n+m} \\
&= (\sqrt{-1})^{\frac{m(m-1)}{2}} \Big(\sum_{1\le r\le m} p_{n+r,ji} (-1)^{r-1} e_j e_{n+1} \cdots \hat{e}_{n+r} \cdots e_{n+m} \\
&\quad - \sum_{1\le r<s\le m} \omega_{n+s,n+r}(e_i) e_{n+1} \cdots \overset{(r)}{e}_{n+s} \cdots \overset{(s)}{e}_{n+s} \cdots e_{n+m} \\
&\quad - \sum_{1\le s<r\le m} \omega_{n+s,n+r}(e_i) e_{n+1} \cdots \overset{(s)}{e}_{n+s} \cdots \overset{(r)}{e}_{n+s} \cdots e_{n+m} \Big) \\
&= (\sqrt{-1})^{\frac{m(m-1)}{2}} \Big(\sum_{1\le r\le m} p_{n+r,ji} (-1)^{r-1} e_j e_{n+1} \cdots \hat{e}_{n+r} \cdots e_{n+m} \\
&\quad + \sum_{1\le r<s\le m} (-1)^{s-r} \omega_{n+s,n+r}(e_i) e_{n+1} \cdots \hat{e}_{n+r} \cdots \hat{e}_{n+s} \cdots e_{n+m} \\
&\quad + \sum_{1\le s<r\le m} (-1)^{r-s} \omega_{n+s,n+r}(e_i) e_{n+1} \cdots \hat{e}_{n+s} \cdots \hat{e}_{n+r} \cdots e_{n+m} \Big) \\
&= (\sqrt{-1})^{\frac{m(m-1)}{2}} \sum_{1\le r\le m} p_{n+r,ji} (-1)^{r-1} e_j e_{n+1} \cdots \hat{e}_{n+r} \cdots e_{n+m} \\
&= p_{Aji} e_j e_A \omega.
\end{aligned}$$

□

Lemma 2.3. *Suppose that M^n is a spin submanifold whose normal bundle is also spin. The connection $\overline{\nabla}$ is compatible with $\langle\ ,\ \rangle$.*

Proof. Since $\omega e_j e_A = -e_j e_A \omega$ for any m, we obtain

$$\begin{aligned}
e_i \langle \varphi, \psi \rangle &= e_i(\omega\varphi, \psi) \\
&= (\widetilde{\nabla}_i \omega \varphi, \psi) + (\omega \widetilde{\nabla}_i \varphi, \psi) + (\omega\varphi, \widetilde{\nabla}_i \psi) \\
&= (p_{Aij} e_j e_A \omega \varphi, \psi) + \langle \overline{\nabla}_i \varphi, \psi \rangle + \frac{1}{2} (p_{Aij} \omega e_j e_A \varphi, \psi) \\
&\quad + \langle \varphi, \overline{\nabla}_i \psi \rangle + \frac{1}{2} (\omega\varphi, p_{Aij} e_j e_A \psi) \\
&= \langle \overline{\nabla}_i \varphi, \psi \rangle + \langle \varphi, \overline{\nabla}_i \psi \rangle.
\end{aligned}$$

□

Let N^{n+m} be an $(n+m)$-dimensional pseudo-Riemannian manifold and M^n an n-dimensional spin spacelike submanifold of N^{n+m}. Suppose that the normal bundle of M^n is also spin. The Dirac-Witten operator over M^n is defined as

$$\widetilde{D} = \sum_i e_i \widetilde{\nabla}_i.$$

The intrinsic Dirac operator of M^n acting on $\mathbb{S}$ is defined as

$$\overline{D} = \sum_i e_i \overline{\nabla}_i.$$

The relation between the operators $\widetilde{D}$ and $\overline{D}$ is given by

$$\widetilde{D} = e_i(\overline{\nabla}_i + \frac{1}{2} p_{Aij} e_j e_A) = \overline{D} - \frac{1}{2} P_A e_A$$

where $P_A = \sum_{i=1}^n p_{Aii}$.

Proposition 2.4. *Suppose that M^n is a compact manifold without boundary.*

(i) *The Dirac operator $\overline{D}$ is formally self-adjoint with respect to the positive definite L^2 inner product $\int_M \langle \ , \ \rangle$;*

(ii) *The Dirac-Witten operator $\widetilde{D}$ is formally self-adjoint with respect to this inner product if the codimension of M^n is odd.*

Proof. A straightforward calculation yields

$$\begin{aligned} e_i \langle e_i \varphi, \psi \rangle &= \langle \overline{\nabla}_i e_i \varphi, \psi \rangle + \langle e_i \overline{\nabla}_i \varphi, \psi \rangle + \langle e_i \varphi, \overline{\nabla}_i \psi \rangle \\ &= \langle \overline{\nabla}_i e_i \varphi, \psi \rangle + \langle \overline{D} \varphi, \psi \rangle - \langle \varphi, \overline{D} \psi \rangle. \end{aligned}$$

Define the vector field $X = \langle e_i \varphi, \psi \rangle e_i$, then

$$\begin{aligned} \mathrm{div} X &= \overline{g}(\overline{\nabla}_i X, e_i) \\ &= \overline{g}\left(\overline{\nabla}_i(\langle e_j \varphi, \psi \rangle e_j), e_i\right) \\ &= e_i \langle e_i \varphi, \psi \rangle - \langle \overline{\nabla}_i e_i \varphi, \psi \rangle \\ &= e_i \langle e_i \varphi, \psi \rangle - \langle \overline{\nabla}_i e_i \varphi, \psi \rangle. \end{aligned}$$

Therefore the Dirac operator $\overline{D}$ is formally self-adjoint. By the relation between $\overline{D}$ and $\widetilde{D}$, it follows

$$\mathrm{div} X = \langle \widetilde{D} \varphi, \psi \rangle - \langle \varphi, \widetilde{D} \psi \rangle + \frac{P_A}{2} \langle e_A \varphi, \psi \rangle - \frac{P_A}{2} \langle \varphi, e_A \psi \rangle.$$

For m odd, the sum of the last two terms vanishes, which implies that $\widetilde{D}$ is formally self-adjoint. $\square$

Now we derive the following Weitzenböck type formula.

Theorem 2.5. *If the codimension of M^n is odd, then*

$$\widetilde{D}^2 = \widetilde{\nabla}^* \widetilde{\nabla} + \frac{1}{4} \sum_{i,j} \widetilde{R}_{ijij} - \frac{1}{2} \sum_{i,j,A} \widetilde{R}_{ijAj} e_i e_A + \frac{1}{4} \sum_{\{\alpha,\beta\} \neq \{i,A\}; \alpha < \beta} \widetilde{R}_{iA\alpha\beta} e_i e_A e_\alpha e_\beta, \quad (3)$$

where $\widetilde{\nabla}_j^* = -\widetilde{\nabla}_j + p_{Aij} e_i e_A \varphi$ is the formal adjoint of $\widetilde{\nabla}$.

Proof. It is straightforward that

$$\begin{aligned}
\widetilde{D}^2\varphi =& e_i\widetilde{\nabla}_i(e_j\widetilde{\nabla}_j\varphi) \\
=& e_i\widetilde{\nabla}_i e_j\widetilde{\nabla}_j\varphi + e_ie_j\widetilde{\nabla}_i\widetilde{\nabla}_j\varphi \\
=& p_{Aij}e_ie_A\widetilde{\nabla}_j\varphi + \frac{1}{2}e_ie_j(\widetilde{\nabla}_i\widetilde{\nabla}_j - \widetilde{\nabla}_j\widetilde{\nabla}_i)\varphi - \widetilde{\nabla}_i\widetilde{\nabla}_i\varphi \\
=& \widetilde{\nabla}^*\widetilde{\nabla}\varphi - \frac{1}{8}\widetilde{R}_{ij\alpha\beta}e_ie_je_\alpha e_\beta\varphi \\
=& \widetilde{\nabla}^*\widetilde{\nabla}\varphi - \frac{1}{8}\sum_{\gamma\neq\alpha\neq\beta}\widetilde{R}_{i\gamma\alpha\beta}e_ie_\gamma e_\alpha e_\beta\varphi - \frac{1}{8}\widetilde{R}_{i\alpha\alpha\beta}e_ie_\alpha e_\alpha e_\beta\varphi \\
&- \frac{1}{8}\widetilde{R}_{i\beta\alpha\beta}e_ie_\beta e_\alpha e_\beta\varphi + \frac{1}{8}\widetilde{R}_{iA\alpha\beta}e_ie_Ae_\alpha e_\beta\varphi \\
=& \widetilde{\nabla}^*\widetilde{\nabla}\varphi - \frac{1}{4}\widetilde{R}_{i\beta\alpha\beta}e_ie_\alpha g^{\beta\beta}\varphi + \frac{1}{8}\widetilde{R}_{iAi\beta}e_ie_Ae_ie_\beta\varphi \\
&+ \frac{1}{8}\widetilde{R}_{iAA\beta}e_ie_Ae_Ae_\beta\varphi + \frac{1}{8}\sum_{\{\alpha,\beta\}\neq\{i,A\}}\widetilde{R}_{iA\alpha\beta}e_ie_Ae_\alpha e_\beta\varphi \\
=& \widetilde{\nabla}^*\widetilde{\nabla}\varphi - \frac{1}{4}\widetilde{R}_{i\beta\alpha\beta}e_ie_\alpha g^{\beta\beta}\varphi \\
&- \frac{1}{8}\widetilde{R}_{iAi\beta}e_Ae_\beta g^{ii}\varphi + \frac{1}{8}\widetilde{R}_{BiB\beta}e_ie_\beta g^{BB}\varphi \\
&+ \frac{1}{8}\sum_{\{\alpha,\beta\}\neq\{i,A\}}\widetilde{R}_{iA\alpha\beta}e_ie_Ae_\alpha e_\beta\varphi \\
=& \widetilde{\nabla}^*\widetilde{\nabla}\varphi + \frac{1}{4}\widetilde{R}_{i\beta i\beta}g^{ii}g^{\beta\beta}\varphi - \frac{1}{4}\widetilde{R}_{i\beta A\beta}e_ie_Ag^{\beta\beta}\varphi \\
&- \frac{1}{8}\widetilde{R}_{iAiA}g^{AA}g^{ii}\varphi - \frac{1}{4}\widetilde{R}_{iAij}e_Ae_jg^{ii}\varphi \\
&- \frac{1}{8}\widetilde{R}_{BiBi}g^{ii}g^{BB}\varphi - \frac{1}{4}\widetilde{R}_{BiBA}e_ie_Ag^{BB}\varphi \\
&+ \frac{1}{8}\sum_{\{\alpha,\beta\}\neq\{i,A\}}\widetilde{R}_{iA\alpha\beta}e_ie_Ae_\alpha e_\beta\varphi \\
=& \widetilde{\nabla}^*\widetilde{\nabla}\varphi + \frac{1}{4}\sum_{i,j}\widetilde{R}_{ijij}\varphi - \frac{1}{2}\sum_{i,j,A}\widetilde{R}_{ijAj}e_ie_A\varphi \\
&+ \frac{1}{4}\sum_{\{\alpha,\beta\}\neq\{i,A\};\alpha<\beta}\widetilde{R}_{iA\alpha\beta}e_ie_Ae_\alpha e_\beta\varphi.
\end{aligned}$$

□

Gauss and Codazzi equations (2) imply

$$\sum_{i,j}\widetilde{R}_{ijij} = \mu, \qquad \sum_j\widetilde{R}_{ijAj} = \varpi_{iA}$$

where

$$\mu = \frac{1}{2}\left(\overline{R} + (\operatorname{Tr} p_A)^2 - |p_A|^2\right), \qquad \varpi_{iA} = \overline{\nabla}^j p_{Aji} - \overline{\nabla}_i \operatorname{Tr} p_A.$$

Therefore (3) can be written as

$$\widetilde{D}^2 = \widetilde{\nabla}^*\widetilde{\nabla} + \frac{1}{2}\Big(\mu - \sum_{i,A} \varpi_{iA} e_i e_A\Big) + \frac{1}{4} \sum_{\{\alpha,\beta\}\neq\{i,A\};\alpha<\beta} \widetilde{R}_{iA\alpha\beta} e_i e_A e_\alpha e_\beta. \tag{4}$$

We say that M^n satisfies *the generalized dominant energy condition* if

$$\mu \geq U := \sqrt{\sum_{i,A} \varpi_{iA}^2} + \frac{1}{2}\sqrt{\sum_{\{\alpha,\beta\}\neq\{i,A\};\alpha<\beta} \widetilde{R}^2_{iA\alpha\beta}}. \tag{5}$$

Theorem 2.6. *Let M^n be a compact spacelike spin submanifold of a pseudo-Riemannian manifold N^{n+m}. Suppose that the normal bundle of M^n is spin and odd-dimensional. Let λ be any eigenvalue of the Dirac-Witten operator $\widetilde{D}$ with a corresponding eigenspinor ϕ. If the generalized dominant energy condition (5) holds, then*

$$\lambda^2 \geq \frac{n}{2(n-1)} \inf_M \big(\mu - U\big). \tag{6}$$

Proof. Note that (4) gives

$$\int_M |\widetilde{D}\varphi|^2 \geq \int_M |\widetilde{\nabla}\varphi|^2 + \frac{1}{2}\left(\mu - U\right)|\varphi|^2. \tag{7}$$

Define the modified connection [10]

$$\widetilde{\nabla}_i^\lambda = \widetilde{\nabla}_i + \frac{\lambda}{n} e_i.$$

For the eigenspinor φ corresponding to the eigenvalue λ, we have

$$|\widetilde{\nabla}^\lambda \varphi|^2 = |\widetilde{\nabla}\varphi|^2 - \frac{\lambda^2}{n}|\varphi|^2.$$

This together with (7) implies (6). If equality holds, then (7) implies that $\widetilde{\nabla}^\lambda\varphi = 0$ and $\mu - U$ is a non-negative constant.

□

3. Boundary value problems

Let F be a Hermitian vector bundle over a Riemannian manifold M^n with nonempty boundary Σ and let $\mathcal{D}$ be a first order elliptic operator acting on the vector bundle F. An elliptic boundary condition for $\mathcal{D}$ can be defined as follows.

The Calderón projector is defined as

$$\mathcal{P}_+(\mathcal{D}) : H^{\frac{1}{2}}(F|_\Sigma) \longrightarrow \big\{\psi|_\Sigma : \psi \in H^1(F), \mathcal{D}\psi = 0\big\}$$

where H^s is the Sobolev space. It is well known that $\mathcal{P}_+(\mathcal{D})$ is a pseudo-differential operator of order zero. Although the Calderón projector is not unique, its principal symbol $\mathbf{p}_+(\mathcal{D})$ is uniquely determined by the principal symbol $\sigma_{\mathcal{D}}$

$$\mathbf{p}_+(\mathcal{D})(u) = -\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma}\left[(\sigma_{\mathcal{D}}(\nu))^{-1}\sigma_{\mathcal{D}}(u) - \zeta I\right]^{-1} d\zeta \tag{8}$$

for any point $p \in \Sigma$ and $u \in T_p\Sigma$, where ν is the inner unit normal along the boundary Σ and Γ is any simple closed contour oriented clockwise and enclosing all poles of the integrand in $\Im\zeta < 0$. Then, an elliptic boundary condition can be defined in terms of $\mathbf{p}_+(\mathcal{D})$ [22, 7]. We refer to [4, 1] for the general discussion of the boundary value problems for Dirac-type operators.

Definition 3.1. A pseudo-differential operator

$$B : L^2(F|_\Sigma) \longrightarrow L^2(V)$$

where $V \longrightarrow \Sigma$ is a complex vector bundle over the boundary, is called a (global) elliptic boundary condition if its principal symbol

$$b : T\Sigma \longrightarrow \mathrm{Hom}_{\mathbb{C}}(F|_\Sigma, V)$$

satisfies, for any non-trivial $u \in T_p\Sigma, p \in \Sigma$, that the restriction

$$b(u)|_{\text{Image } \mathbf{p}_+(\mathcal{D})(u)} : \text{Image } \mathbf{p}_+(\mathcal{D})(u) \subset F_p \longrightarrow V_p$$

is an isomorphism onto the image $b(u) \subset V_p$. Moreover, if rank $V =$ dim Image $\mathbf{p}_+(\mathcal{D})(u)$, we say that B is a *local elliptic boundary condition.*

In this case we say that

$$\begin{cases} \mathcal{D}\psi = \chi & \text{in } M^n, \\ B\psi|_\Sigma = \phi & \text{on } \Sigma \end{cases} \tag{EBP}$$

is an elliptic boundary problem. An elliptic boundary problem such as (EBP) has a solution $\psi \in H^1(F)$ for any pair (χ, ϕ) in a subspace of $L^2(F) \times H^{1/2}(V)$ of finite codimension. Moreover, this solution is unique up to a finite-dimensional kernel, i.e., (EBP) is of Fredholm type.

Now we use the argument in [15, 13] to study an elliptic boundary problem on the spinor bundle. Let M^n be a compact spacelike spin submanifold of pseudo-Riemannian manifold N^{n+m} with spin normal bundle. Suppose M^n has a nonempty boundary Σ endowed with an induced Riemannian and spin structures. For any $p \in \Sigma$, we choose the orthonormal basis $\{e_\alpha\}_{\alpha=1}^{n+m}$ such that e_i $(1 \le i \le n)$ tangent to M^n with e_n the outer normal to the boundary Σ, e_a $(1 \le a \le n-1)$ tangent to Σ and as before $e_A (n+1 \le A \le n+m)$ normal to M^n. Denote the boundary projection operator by

$$B_\pm = \frac{1}{2}\left(I \pm e_n\omega\right).$$

Proposition 3.1. *If m is odd, then the following boundary problem for the Dirac-Witten operator*

$$\begin{cases} \widetilde{D}\psi = \lambda\psi, & \textit{in } M^n, \\ B_\pm\psi|_\Sigma = 0, & \textit{on } \Sigma \end{cases} \tag{9}$$

is elliptic. Moreover, (9) has a discrete spectrum with finite dimensional eigenspaces consisting of smooth spinor fields, unless it is the whole complex plane.

Proof. Without loss of generality, we prove the theorem only for the boundary operator B_+. Obviously, the symbol of the Dirac-Witten operator can be calculated by

$$\sigma_{\widetilde{D}}(v) = \sqrt{-1}v, \qquad \forall\, v \in\ TM^n.$$

By (8), the principal symbol $\mathbf{p}_+(\widetilde{D})$ of the Calderón projector of the Dirac-Witten operator is given by

$$\mathbf{p}_+(\widetilde{D})(u) = -\frac{1}{2|u|}\left(\sqrt{-1}(-e_n)u - |u|I\right), \qquad \text{for } u \in T\Sigma.$$

¿From Proposition 1 in [13], we have

$$\text{Image } \mathbf{p}_+(\widetilde{D})(u) = \left\{\psi \in \mathbb{S} : \sqrt{-1}(-e_n)u\psi = -|u|\psi\right\},$$

$$\dim \text{ Image}\mathbf{p}_+(\widetilde{D})(u) = \frac{1}{2}\dim \mathbb{S}.$$

Since the eigenspaces of $e_n\omega$ corresponding to the eigenvalues 1 and -1 are interchanged by e_n, we have

$$\text{rank } V = \frac{1}{2}\dim \mathbb{S}. \tag{10}$$

Since B_+ is a pseudo-differential operator of order zero, its principal symbol $b_+(u)$, on each vector $u \in T\Sigma$, coincides with the operator itself, that is,

$$b_+(u) = \frac{1}{2}(I + e_n\omega), \qquad \forall\ u \in T\Sigma.$$

If $b_+(u)\varphi = 0$, i.e. $e_n\omega\varphi = -\varphi$, then

$$\begin{aligned}\sqrt{-1}e_nu\varphi &= -\sqrt{-1}e_nu(e_n\omega)\varphi \\ &= e_n\omega(\sqrt{-1}e_n\omega\varphi).\end{aligned}$$

This implies that $\sqrt{-1}e_nu\varphi$ belongs to the positive eigenspace of $e_n\omega$. Therefore,

$$\text{Ker } b_+(u) \cap \text{Image } \mathbf{p}_+(\widetilde{D})(u) = \{0\}. \tag{11}$$

¿From (10) and (11), the elliptic boundary condition for the pseudo-differential operator B_+ are satisfied by Proposition 1 in [13]. The remaining assertions on eigenvalues and eigenspaces are straightforward (see [7, 16]). □

For any point $p \in \Sigma$ and an orthonormal basis $\{e_i\}$ of T_pM^n with e_n the outward normal to Σ and e_a tangent to Σ for $1 \leq a \leq n-1$. Let

$$h_{ab} = \overline{g}(\overline{\nabla}_a e_n, e_b)$$

be the components of the second fundamental form of Σ. Let $H = \sum_{a=1}^{n-1} h_{aa}$ be its mean curvature.

Denote by ∇ the lift of the Levi-Civita connection of Σ to the spinor bundle $\mathbb{S}|_\Sigma$. Then the spinorial Gauss formula is given by

$$\overline{\nabla}_a = \nabla_a + \frac{1}{2}h_{ab}e_n e_b.$$

The Dirac operator D of Σ acting on $\mathbb{S}|_\Sigma$ is defined as

$$D = e_a \nabla_a.$$

A straightforward calculation yields the connection ∇ is also compatible with the Hermitian metric $\langle\ ,\ \rangle$ on the spinor bundle over the boundary Σ. Moreover,

$$\widetilde{\nabla}_a = \nabla_a + \frac{1}{2}h_{ab}e_n e_b + \frac{1}{2}p_{Aaj}e_j e_A.$$

Lemma 3.2. *The following identities hold*

$$\nabla_a(e_n\varphi) = e_n\nabla_a\varphi, \qquad \nabla_a(e_A\varphi) = e_A\nabla_a\varphi.$$

Proof.

$$\begin{aligned}
\nabla_a(e_n\varphi) &= \left(\overline{\nabla}_a - \frac{1}{2}h_{ab}e_n e_b\right)(e_n\varphi) \\
&= \overline{\nabla}_a(e_n\varphi) - \frac{1}{2}h_{ab}e_n e_b e_n\varphi \\
&= \overline{\nabla}_a e_n\varphi + e_n\overline{\nabla}_a\varphi - \frac{1}{2}h_{ab}e_b\varphi \\
&= h_{ab}e_b\varphi + \left(e_n\nabla_a + \frac{1}{2}h_{ab}e_n e_b\right)\varphi - \frac{1}{2}h_a b e_b\varphi \\
&= e_n\nabla_a\varphi. \\
\nabla_a(e_A\varphi) &= \left(\widetilde{\nabla}_a - \frac{1}{2}h_{ab}e_n e_b - \frac{1}{2}p_{Baj}e_j e_B\right)(e_A\varphi) \\
&= \widetilde{\nabla}_a(e_A\varphi) - \frac{1}{2}h_{ab}e_n e_b e_A\varphi - \frac{1}{2}p_{Baj}e_j e_B e_A\varphi \\
&= p_{Aaj}e_j\varphi + e_A\left(\nabla_a + \frac{1}{2}h_{ab}e_n e_b + \frac{1}{2}p_{Baj}e_j e_B\right)\varphi \\
&\quad - \frac{1}{2}h_{ab}e_n e_b e_A\varphi - \frac{1}{2}p_{Baj}e_j e_B e_A\varphi \\
&= e_A\nabla_a\varphi.
\end{aligned}$$

$\square$

Lemma 3.3. *The Dirac operator acting on the spinor bundle $\mathbb{S}$ and the local boundary condition satisfy the following relations*

$$e_n D B_\pm = B_\mp e_n D, \qquad e_A e_a B_\pm = B_\mp e_A e_a.$$

where $B_\pm = \frac{1}{2}(I \pm e_n\omega)$ are the projection operators acting on the spinor bundle $\mathbb{S}$.

Proof. ¿From Lemma 3.2, we get

$$\begin{aligned}
e_n D B_\pm \varphi &= \frac{1}{2} e_n D(\varphi \pm e_n \omega \varphi) \\
&= \frac{1}{2} e_n D\varphi \pm \frac{1}{2} e_n D(e_n \omega \varphi) \\
&= \frac{1}{2} e_n D\varphi \pm \frac{1}{2} e_n e_a \nabla_a (e_n \omega \varphi) \\
&= \frac{1}{2} e_n D\varphi \pm \frac{1}{2} e_n e_a e_n \nabla_a (\omega \varphi) \\
&= \frac{1}{2} e_n D\varphi \pm \frac{1}{2} e_n e_a e_n \omega \nabla_a \varphi \\
&= \frac{1}{2} e_n D\varphi \pm \frac{1}{2} e_n \omega e_a e_n \nabla_a \varphi \\
&= \frac{1}{2} e_n D\varphi \mp \frac{1}{2} e_n \omega e_n D\varphi \\
&= B_\mp e_n D\varphi
\end{aligned}$$

and

$$\begin{aligned}
e_A e_a B_\pm \varphi &= \frac{1}{2} e_A e_a (\varphi \pm e_n \omega \varphi) \\
&= \frac{1}{2} e_A e_a \varphi \pm \frac{1}{2} e_A e_a e_n \omega \varphi \\
&= \frac{1}{2} (e_A e_a \varphi \mp e_n \omega e_A e_a \varphi) \\
&= B_\mp e_A e_a \varphi.
\end{aligned}$$

□

Lemma 3.4. *Let M^n be a compact spacelike spin submanifold of a pseudo-Riemannian manifold N^{n+m}. Suppose that the normal bundle of M is spin and m is odd. Let $\widetilde{D}$ be the Dirac-Witten operator on the spinor bundle over M. Denote*

$$\mathbf{R}_M = \mu - \varpi_{iA} e_i e_A + \frac{1}{2} \sum_{\{\alpha,\beta\} \neq \{i,A\}; \alpha < \beta} \widetilde{R}_{iA\alpha\beta} e_i e_A e_\alpha e_\beta. \tag{12}$$

If M^n has a nonempty boundary Σ, then

$$\int_\Sigma \langle e_n D\varphi, \varphi\rangle - \frac{H}{2}\langle \varphi, \varphi\rangle + \frac{\mathrm{Tr}\,(p_A|_\Sigma)}{2}\langle e_A e_n \varphi, \varphi\rangle - \frac{p_{Aan}}{2}\langle e_A e_a \varphi, \varphi\rangle$$
$$= \int_M |\widetilde{\nabla}\varphi|^2 + \frac{1}{2}\langle \mathbf{R}_M \varphi, \varphi\rangle - |\widetilde{D}\varphi|^2.$$

Proof. A straightforward calculation yields

$$\int_\Sigma \langle e_n \widetilde{D}\varphi, \varphi\rangle = \int_M \langle \widetilde{D}^2 \varphi, \varphi\rangle - \int_M |\widetilde{D}\varphi|^2,$$
$$\int_\Sigma \langle \widetilde{\nabla}_n \varphi, \varphi\rangle = -\int_M \langle p_{Aij} e_j e_A \widetilde{\nabla}_i \varphi, \varphi\rangle + \int_M \langle \widetilde{\nabla}\widetilde{\nabla}\varphi, \varphi\rangle + \int_M |\widetilde{\nabla}\varphi|^2.$$

Thus

$$\begin{aligned}
\int_M |\widetilde{D}\varphi|^2 - &\frac{1}{2}\langle \mathbf{R}_M\varphi,\varphi\rangle - |\widetilde{\nabla}\varphi|^2 \\
&= -\int_\Sigma \langle \widetilde{\nabla}_n\varphi + e_n\widetilde{D}\varphi,\varphi\rangle \\
&= -\int_\Sigma \langle \widetilde{\nabla}_n\varphi + e_n e_i\widetilde{\nabla}_i\varphi,\varphi\rangle \\
&= -\int_\Sigma \langle e_n e_a\widetilde{\nabla}_a\varphi,\varphi\rangle \\
&= -\int_\Sigma \langle e_n e_a(\nabla_a + \frac{1}{2}h_{ab}e_n e_b + \frac{1}{2}p_{Aaj}e_j e_A)\varphi,\varphi\rangle \\
&= -\int_\Sigma \langle (e_n D - \frac{1}{2}H)\varphi,\varphi\rangle - \frac{1}{2}\int_\Sigma \langle \mathrm{Tr}\,(p_A|_\Sigma)e_A e_n\varphi,\varphi\rangle \\
&\quad + \frac{1}{2}\int_\Sigma \langle p_{Aan}e_A e_a\varphi,\varphi\rangle.
\end{aligned}$$

□

Denote by $\vec{P} = \mathrm{Tr}\,(p_A|_\Sigma)e_A$ the restriction of the mean curvature vector on Σ. Then Σ is a *generalized future/past apparent horizon* if

$$H \mp \vec{P}\omega \geq 0 \tag{13}$$

on Σ as an endomorphism over $\mathbb{S}$. The physical explanation of this definition relies on the behavior of null geodesics in this case. This will be examined elsewhere.

Theorem 3.5. *Let M^n be a compact spacelike spin submanifold of a pseudo-Riemannian manifold N^{n+m}. Suppose that the normal bundle of M is spin and m is odd. Let λ be an eigenvalue of the Dirac-Witten operator with a corresponding spinor φ. If M^n has a nonempty boundary Σ which is a future or past apparent horizon, and the generalized dominant energy condition (5) holds, then, under the local boundary condition $B_-\varphi = 0$ for future apparent horizon, or $B_+\varphi = 0$ for past apparent horizon, we have*

$$\lambda^2 \geq \frac{n}{2(n-1)} \inf_M \big(\mu - U\big). \tag{14}$$

Proof. For the local boundary condition, we have

$$\int_\Sigma \langle e_n D\varphi,\varphi\rangle = 0, \qquad \int_\Sigma \langle p_{Aan}e_A e_a\varphi,\varphi\rangle = 0.$$

The theorem follows by using the modified connection $\widetilde{\nabla}_i^\lambda = \widetilde{\nabla}_i + \frac{\lambda}{n}e_i$. □

Remark 3.1. If $m = 1$, Theorem 2.6 and Theorem 3.5 reduce to the main estimates in [19]. We can also apply the conformal methods in [12, 19] to estimate the eigenvalues of the submanifold Dirac-Witten operator. Recall, in the case where $m = 1$ [19], the Einstein tensors of the conformal metric $\widetilde{\mathbf{g}} = f^{\frac{4}{n-2}}\widetilde{g}$ with $f > 0$, $df(e_n) = 0$ satisfy

$$\mathbf{T}_{nn} = T_{nn} + \frac{2(n-1)}{n-2} f^{-1} \triangle f, \qquad \mathbf{T}_{an} = T_{an} + (n-1) h_a^b \nabla_b u$$

where $f^{\frac{4}{n-2}} = e^{2u}$ (*The authors would like to thank Daniel Maerten who pointed out an error in* [19] *where the term* $h_a^b \nabla_b u$ *was missing in the second above formula*). For an eigenspinor ϕ, we define $T_\phi^a = \langle \phi, e_n e_a \phi \rangle / |\phi|^2$ on the complement of the zero's set of ϕ, and by zero on its zero set. Let

$$L_\phi = \frac{4(n-1)}{n-2}\Big(\triangle + T_\phi^a h_a^b \nabla_b\Big) + 2\Big(T_{nn} - \sqrt{\sum_a T_{an}^2}\Big).$$

Then the operator L should be replaced by L_ϕ in Theorem 3 and Theorem 6 in [19] (It would be interesting to discuss other types of boundary conditions [14, 13, 8] and to generalize these results to the case $m > 1$).

4. Application to gravity

The Dirac-Witten operator is closely related to the total energy-momentum of spacetimes [26]. A similar result is generalized to 5-dimensional spacetime N^{4+1} [28, 30], and to arbitary higher dimensional spacetime N^{n+1} [9]. It is hence natural to study the Dirac-Witten operator in the set-up of pseudo-Riemannian manifolds N^{n+m} with $m \geq 1$.

Let N^{n+m} be a pseudo-Riemannian manifold. Let M^n be a spacelike submanifold of N^{n+m} with the induced Riemannian metric denoted by g (instead of $\overline{g}$ for simplicity) and second fundamental form p_A. An initial data set (M, g, p_A) is asymptotically flat if there is a compact set K such that $M \setminus K$ is the disjoint union of a finite number of subsets $M_1, \cdots, M_l$ - called the "ends" of M - each diffeomorphic to $\mathbb{R}^n \setminus B_r^n$, where B_r^n is the closed ball of radius r with center at the coordinate origin. In each end, g and p_A satisfy, as $r \to \infty$,

$$g_{ij} = \delta_{ij} + O\big(\frac{1}{r^{n-2}}\big), \qquad \partial_k g_{ij} = O\big(\frac{1}{r^{n-1}}\big), \qquad \partial_l \partial_k g_{ij} = O\big(\frac{1}{r^n}\big),$$

$$p_{Aij} = O\big(\frac{1}{r^{n-1}}\big), \qquad \partial_k p_{Aij} = O\big(\frac{1}{r^n}\big)$$

where $\{x^i\}$ are the Euclidean coordinates of $\mathbb{R}^n$. Moreover, the scalar curvature R of M is assumed to be in $L^1(M)$.

The ADM total energy E_l and the generalized total linear momentum P_{lkA} of the end M_l are defined as follows

$$\begin{aligned} E_l =& \frac{1}{4\mathrm{Vol}(S^{n-1})} \lim_{r\to\infty} \int_{S_r^{n-1}} (\partial_j g_{ij} - \partial_i g_{jj}) * dx^i, \\ P_{lkA} =& \frac{1}{2\mathrm{Vol}(S^{n-1})} \lim_{r\to\infty} \int_{S_r^{n-1}} (p_{Aki} - \delta_{ki} h_{Ajj}) * dx^i, \end{aligned}$$

where S_r^{n-1} is the sphere of radius r in the end and S^{n-1} is the unit sphere in the n-dimensional Euclidean space. The generalized total linear momentum P_l of the end M_l is actually a map from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}$, with the components P_{lkA} where $1 \leq k \leq n$, $n+1 \leq A \leq n+m$. It is therefore a bi-vector. Now we can prove the following generalized positive mass theorem.

Theorem 4.1. *Let M^n be a compact spacelike spin submanifold of a pseudo-Riemannian manifold N^{n+m}, which has possibly finite number of generalized future/past apparent horizons Σ_i. Suppose that the normal bundle of M is spin and m is odd. If the generalized dominant energy condition (5) holds, then*

$$E_l \geq \sqrt{\sum_{k,A} P_{lkA}^2}. \tag{15}$$

That $E_{l_0} = 0$ for some end M_{l_0} implies that M has only one end and $\widetilde{R}_{ij\alpha\beta} = 0$ over M.

Proof. Let $\breve{\varphi}$ be any constant spinor on the flat space $\mathbb{R}^{n+m}$ with unit norm under the positive definite spinor inner product. Then we can solve the boundary value problem for the Dirac-Witten operator on M using the same argument as that in [26, 21, 3, 28, 29, 30, 9]. Let φ be such a solution with the boundary value $\breve{\varphi}$ at infinity on the end M_l, and zero at infinity on the other ends. Denote ∂_α by $\breve{e}_\alpha$. The formula (4) implies that

$$\begin{aligned}
\int_M |\widetilde{\nabla}\varphi|^2 \leq & \int_{S_\infty^{n-1}} \langle \breve{\varphi}, \sum_{i\neq j} \breve{e}_i \breve{e}_j \widetilde{\nabla}_j \breve{\varphi}\rangle * dx^i \\
=& 4E_l + \int_{S_\infty^{n-1}} \langle \varphi, \frac{1}{2} \sum_{i\neq j} p_{Ajk} \breve{e}_i \breve{e}_j \breve{e}_k \breve{e}_A \breve{\varphi}\rangle * dx^i \\
=& 4E_l + \int_{S_\infty^{n-1}} \langle \varphi, \frac{1}{2} \left(p_{Aik} - \delta_{ik} p_{Ajj}\right) \breve{e}_k \breve{e}_A \breve{\varphi}\rangle * dx^i \\
=& 4E_l + 4\langle \breve{\varphi}, P_{lkA} \breve{e}_k \breve{e}_A \breve{\varphi}\rangle.
\end{aligned}$$

Then (15) follows by choosing $\breve{\varphi}$ such that $P_{lkA}\breve{e}_k\breve{e}_A\breve{\varphi} = -\sqrt{\sum_{k,A} P_{lkA}^2}\,\breve{\varphi}$.

If $E_{l_0} = 0$, then for any constant spinor $\breve{\varphi}$, we have $\widetilde{\nabla}_i \varphi = 0$ for solution of the Dirac-Witten equation with the boundary value $\breve{\varphi}$ at infinity on the end M_{l_0}, and zero at infinity on the other ends. This implies that M has only one end and $\widetilde{R}_{ij\alpha\beta} = 0$ over M. □

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